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Global degenerating families with periodic monodromies

By

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Abstract

We prove that, given N powers of one surface periodic mapping class, if their composition coincides with the identity, then there exists a degenerating family of Riemann surfaces over a Riemann surface of arbitrary genus k with N singular fibers whose local monodromies correspond to the given N mapping classes respectively. We also show that, in the case $(N, k) = (2, 0)$, the degenerating family constructed by our method has n (-1)-sections if the given mapping classes have n fixed points.

§ 1. Introduction

It is known that the topological types of minimal degenerations of Riemann surfaces of genus at least two are in a bijective correspondence with the conjugacy classes in the surface mapping class group represented by pseudo-periodic maps of negative twist, via topological monodromy. Earle-Sipe [1] and Shiga-Tanigawa [6] showed that the topological monodromy of any degeneration is represented by a pseudo-periodic map of negative twist. The converse of this result was demonstrated by Matsumoto-Montesinos [4]. Namely, given a pseudo-periodic mapping class f of negative twist, they construct a degeneration with singular fiber whose topological monodromy coincides with $[f]$ up to conjugacy. Their argument is quite topological, based on *open book construction.* On the other hand, Takamura [7] gave another algebro-geometric construction, called *cyclic quotient construction.*

In this paper, we consider global cases. A *global degenerating family* (or simply, *degenerating family*) of Riemann surfaces of genus $g \geq 1$ is a proper surjective holomorphic map $\pi : M \to S$ from a compact smooth complex surface M to a compact Riemann

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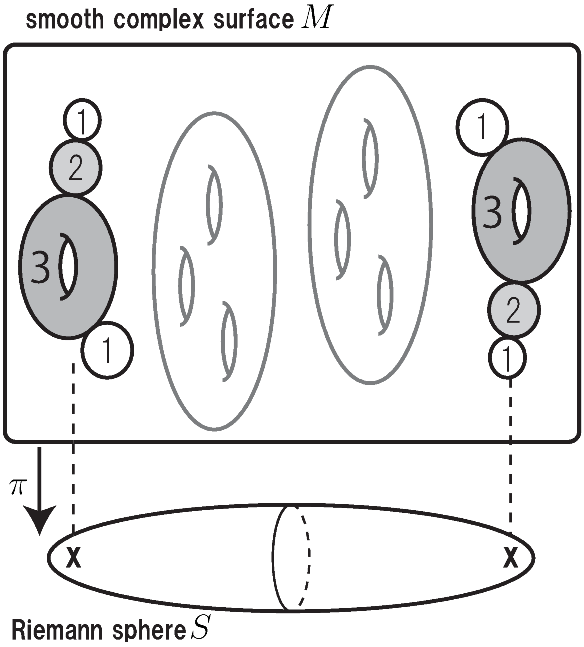

Figure 1. An example of a degenerating family of Riemann surfaces of genus three over the Riemann sphere with two singular fibers.

surface S such that all but finitely many fibers are smooth complex curves of genus g. We call the fibers over the singular values $s_1, s_2, \ldots, s_N \in S$ of π the *singular fibers.* If S is the Riemann sphere, then the composition of the topological monodromies around N singular fibers (precisely, with respect to N loops in S going around each s_j once in the counterclockwise direction with a base point $*$) coincides with the identity. This gives us the following natural problem: *Is there always a degenerating family of Riemann surfaces over the Riemann sphere with N singular fibers whose local monodromies respectively correspond to the given pseudo-periodic mapping classes* $[f_1], [f_2], \ldots, [f_N]$ *of negative twist satisfying* $[f_1] \circ [f_2] \circ \cdots \circ [f_N] = 1$*?*

In Section 2, we begin with the review of cyclic quotient construction for degenerations of Riemann surfaces with periodic monodromy. After that, we give a solution to the above problem for the case $N = 2$. To be precise, we show that, given two *periodic* mapping classes $[f_1]$ and $[f_2]$ satisfying $[f_1] \circ [f_2] = 1$, there exists a degenerating family of Riemann surfaces over the Riemann sphere with two singular fibers whose local monodromies respectively correspond to $[f_1]$ and $[f_2]$ (Proposition 2.5). See Figure 1. Note that two pseudo-periodic mapping classes $[f_1]$ and $[f_2]$ of negative twist are periodic if $[f_1] \circ [f_2] = 1$.

Section 3 is devoted to the review of the total valency for periodic maps. Using this concept, in Section 4, we generalize the statement of Proposition 2.5. We prove that, given N *powers* $[f_1], [f_2], \ldots, [f_N]$ *of one periodic mapping class* $[f]$ satisfying

$[f_1] \circ [f_2] \circ \cdots \circ [f_N] = 1$, there exists a degenerating family of Riemann surfaces over a Riemann surface of *arbitrary genus* k with N singular fibers whose local monodromies respectively correspond to $[f_1], [f_2], \ldots, [f_N]$ (Theorem 4.4). Note that, assuming that $(N, k) = (2, 0)$, we obtain Proposition 2.5. This result may be known to experts, but we can apply this construction to the case for "degenerating families of complex manifolds". See Proposition 4.5.

The total valency also plays a very important role when we clarify the total spaces and the singular fibers of degenerations of Riemann surfaces. See [4], [7]. In the remaining part of this paper (Sections 5–7), by the similar argument to that used in [7], we explicitly describe the degenerating families of Riemann surfaces constructed in Proposition 2.5. In particular, we show that, such a degenerating family has n (-1)-sections (holomorphic sections with self-intersection number -1) if the given periodic mapping classes have n fixed points (Theorem 7.3).

§ 2. Global cyclic quotient construction (special case)

Let us begin with the review of *cyclic quotient construction* for degenerations of Riemann surfaces with periodic monodromies.

Let $\Sigma = \Sigma_g$ be an oriented closed real surface of genus g and let $[f]$ be a periodic mapping class of Σ of order m (so $[f]^m = 1$). By Kerckhoff's theorem [3], there exists a complex structure on Σ and a periodic automorphism of the Riemann surface Σ that represents $[f]$. We denote the automorphism by f again. We consider an automorphism γ of $\Sigma \times \mathbb{C}$ given by

$$\gamma : (x, t) \longmapsto \left(f^{-1}(x), \omega t\right), \tag{2.1}$$

where $\omega := e^{2\pi\sqrt{-1}/m}$. The cyclic group G generated by γ, which is of order m, acts on $\Sigma \times \mathbb{C}$. Then we obtain the quotient space $(\Sigma \times \mathbb{C})/G$ of $\Sigma \times \mathbb{C}$ under the group action of G. Note that the complex surface $(\Sigma \times \mathbb{C})/G$ is not necessarily smooth. In fact, it may possibly have cyclic quotient singularities, which are contained in $(\Sigma \times \{0\})/G$. Let $\mathfrak{r} : M \to (\Sigma \times \mathbb{C})/G$ be the resolution map that resolves all the cyclic quotient singularities of $(\Sigma \times \mathbb{C})/G$ minimally. We next define a holomorphic function $\varphi : \Sigma \times \mathbb{C} \to \mathbb{C}$ by $\varphi(x, t) := t^m$. Since φ is G-invariant, it descends to a holomorphic function $\overline{\varphi} : (\Sigma \times \mathbb{C})/G \to \mathbb{C}$. By construction, we see that general fibers $\overline{\varphi}^{-1}(s)$, $s \neq 0$, are identical to Σ, and that the monodromy automorphism around the central fiber $\overline{\varphi}^{-1}(0)$ $(= (\Sigma \times \{0\})/G)$ coincides with f. Noting that $\mathfrak{r}$ maps $M \setminus \mathfrak{r}^{-1}(\overline{\varphi}^{-1}(0))$ to $((\Sigma \times \mathbb{C})/G) \setminus \overline{\varphi}^{-1}(0)$ isomorphically, we have the following.

Lemma 2.1. *The composition map $\pi := \overline{\varphi} \circ \mathfrak{r} : M \to \mathbb{C}$ is a degeneration of Riemann surfaces of genus g, and the monodromy automorphism around the central fiber $\pi^{-1}(0)$ coincides with f.*

Moreover, we see that the degeneration $\pi : M \to \mathbb{C}$ is linear. (For details, see [7]. We can find the definition of linear degenerations in [8], §15.1.) Thus we obtain Takamura's theorem for periodic case.

Theorem 2.2 ([7]). *For a periodic mapping class* $[f]$ *of* Σ_g $(g \geq 1)$*, there exists a linear degeneration of Riemann surfaces of genus* g *such that the topological monodromy around the singular fiber corresponds to* $[f]$*.*

Before proceeding, we give an alternative definition of the above degeneration $\pi : M \to \mathbb{C}$ as follows. Let $\Phi : \Sigma \times \mathbb{C} \to \mathbb{C}$ denote the second projection. Then Φ is compatible with the G-action, that is, the following diagram is commutative:

$$\begin{array}{ccc} \Sigma \times \mathbb{C} & \xrightarrow{\Phi} & \mathbb{C} \\ \gamma \downarrow & & \downarrow \omega\cdot \\ \Sigma \times \mathbb{C} & \xrightarrow{\Phi} & \mathbb{C} \end{array}$$

Thus Φ determines a holomorphic map $\overline{\Phi} : (\Sigma \times \mathbb{C})/G \to \mathbb{C}/\langle\omega\rangle$. Then we see that $\overline{\Phi} : (\Sigma \times \mathbb{C})/G \to \mathbb{C}/\langle\omega\rangle$ $(\cong \mathbb{C})$ coincides with $\overline{\varphi} : (\Sigma \times \mathbb{C})/G \to \mathbb{C}$. In fact, $\varphi : \Sigma \times \mathbb{C} \to \mathbb{C}$ is factorized as $\varphi = \nu \circ \Phi$, where $\nu : \mathbb{C} \to \mathbb{C}$ is given by $\nu(t) = t^m$. Since the diagram

$$\begin{array}{ccccc} \Sigma \times \mathbb{C} & \xrightarrow{\Phi} & \mathbb{C} & \xrightarrow{\nu} & \mathbb{C} \\ \gamma \downarrow & & \downarrow \omega\cdot & & \| \\ \Sigma \times \mathbb{C} & \xrightarrow{\Phi} & \mathbb{C} & \xrightarrow{\nu} & \mathbb{C} \end{array}$$

is commutative, we have $\overline{\varphi} = \overline{\nu} \circ \overline{\Phi}$, where $\overline{\nu} : \mathbb{C}/\langle\omega\rangle \to \mathbb{C}$ is the holomorphic function determined by ν. Note that $\overline{\nu}$ gives an isomorphism between $\mathbb{C}/\langle\omega\rangle$ and $\mathbb{C}$. Under this identification, $\overline{\Phi} : (\Sigma \times \mathbb{C})/G \to \mathbb{C}$ coincides with $\overline{\varphi} : (\Sigma \times \mathbb{C})/G \to \mathbb{C}$. Then the composition $\overline{\Phi} \circ \mathfrak{r} : M \to \mathbb{C}$ is nothing but the degeneration $\pi : M \to \mathbb{C}$ in Lemma 2.1. Thus we have the following.

Lemma 2.3. *Let* G *be the cyclic group generated by* γ*, where* γ *is an automorphism of* $\Sigma \times \mathbb{C}$ *given by* (2.1)*. Let* $\overline{\Phi} : (\Sigma \times \mathbb{C})/G \to \mathbb{C}/\langle\omega\rangle$ *be the holomorphic map determined by the second projection* $\Phi : \Sigma \times \mathbb{C} \to \mathbb{C}$*, and let* $\mathfrak{r} : M \to (\Sigma \times \mathbb{C})/G$ *be the minimal resolution map of* $(\Sigma \times \mathbb{C})/G$*. Then the composition map* $\pi := \overline{\Phi} \circ \mathfrak{r} : M \to \mathbb{C}/\langle\omega\rangle$ $(\cong \mathbb{C})$ *is a degeneration of Riemann surfaces whose monodromy automorphism coincides with* f*.*

Remark. Note that the base space of the degeneration obtained in Lemma 2.3 is $\mathbb{C}$. For any open disk Δ in $\mathbb{C}$ centered at the origin, we obtain a degeneration of Riemann surfaces over Δ satisfying the same conditions as those of π, by taking the restriction $\pi : \pi^{-1}(\Delta) \to \Delta$ of π.

We now construct a degenerating family of Riemann surfaces of genus g over the Riemann sphere with exactly two singular fibers. We will consider more general case in Section 4. We show the following.

Lemma 2.4. *Let f be a periodic automorphism of a Riemann surface $\Sigma = \Sigma_g$ of genus $g \geq 1$. Then there exists a degenerating family of Riemann surfaces of genus g over the Riemann sphere with two singular fibers whose local monodromies respectively correspond to f and f^{-1}.*

Proof. Let $\mathbb{C}_0$ and $\mathbb{C}_\infty$ be two copies of $\mathbb{C}$, and let S $(= \mathbb{C}P^1)$ be the Riemann sphere with the standard open covering $S = \mathbb{C}_0 \cup \mathbb{C}_\infty$ by identifying $t \in \mathbb{C}_0 \setminus \{0\}$ and $t' \in \mathbb{C}_\infty \setminus \{0\}$ via $t' = 1/t$. We denote $0 \in \mathbb{C}_\infty$ by ∞ as usual. Denote by m the order of f. We consider an automorphism γ of $\Sigma \times S$ given by

$$\begin{cases} \gamma(x,t) = \left(f^{-1}(x), \omega t\right), & (x,t) \in \Sigma \times \mathbb{C}_0, \\ \gamma(x,t') = \left(f^{-1}(x), \omega^{-1} t'\right), & (x,t') \in \Sigma \times \mathbb{C}_\infty, \end{cases}$$

where $\omega := e^{2\pi\sqrt{-1}/m}$. Then the cyclic group G generated by γ acts on $\Sigma \times S$. Note that all the singularities of the complex surface $(\Sigma \times S)/G$, if exist, are cyclic quotient singularities contained in $(\Sigma \times \{0\})/G \sqcup (\Sigma \times \{\infty\})/G$. Let $\mathfrak{r} : M \to (\Sigma \times S)/G$ be the minimal resolution map. We next define two holomorphic functions $\varphi : \Sigma \times \mathbb{C}_0 \to \mathbb{C}_0$ by $\varphi(x,t) := t^m$ and $\varphi' : \Sigma \times \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $\varphi'(x,t') := (t')^m$. They together determine the holomorphic map from $\Sigma \times S$ to S, denoted by φ again. Since $\varphi : \Sigma \times S \to S$ is G-invariant, it descends to a holomorphic map $\overline{\varphi} : (\Sigma \times S)/G \to S$.

We now consider the composition map $\pi := \overline{\varphi} \circ \mathfrak{r} : M \to S$. Set $X_0 := \pi^{-1}(0)$ and $X_\infty := \pi^{-1}(\infty)$. Note that $M \setminus X_\infty = \pi^{-1}(\mathbb{C}_0)$ and $M \setminus X_0 = \pi^{-1}(\mathbb{C}_\infty)$. By Lemma 2.1, the restriction $\pi : \pi^{-1}(\mathbb{C}_0) \to \mathbb{C}_0$ is a degeneration of Riemann surfaces of genus g with singular fiber X_0 whose monodromy automorphism coincides with f. On the other hand, the restriction $\pi : \pi^{-1}(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ is a degeneration of Riemann surfaces with singular fiber X_∞ whose monodromy automorphism coincides with f^{-1}. In fact, we can take γ^{-1} as another generator of G, which acts on $\Sigma \times \mathbb{C}_\infty$ by

$$\gamma^{-1}(x,t') = \left(\left(f^{-1}\right)^{-1}(x), \omega t'\right), \quad (x,t') \in \Sigma \times \mathbb{C}_\infty.$$

Applying Lemma 2.1 to the case f is f^{-1}, we see that the monodromy automorphism around the singular fiber X_∞ of $\pi : \pi^{-1}(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ coincides with f^{-1}. Therefore $\pi : M \to S$ is the desired degenerating family. □

By Lemma 2.4, we have the following.

Proposition 2.5. *Let $[f_1]$ and $[f_2]$ be periodic mapping classes of Σ_g satisfying $[f_1] \circ [f_2] = 1$. Then there exists a degenerating family of Riemann surfaces over the*

Riemann sphere with two singular fibers whose local monodromies respectively correspond to $[f_1]$ *and* $[f_2]$.

§ 3. Total valencies of periodic maps

This section is devoted to review of the total valencies of periodic maps. Let $f : \Sigma \to \Sigma$ be a periodic map of an oriented closed real surface $\Sigma = \Sigma_g$ of genus $g \geq 1$. In what follows, we regard f as a periodic automorphism of a Riemann surface Σ as in the previous section. We denote its order by m.

A point x of Σ is a *multiple point* of f if the least positive integer c that satisfies $f^c(x) = x$ is less than m. In this case, we call c the *recurrence number* of x. Then f^c preserves a small disk centered at x and acts on it as a rotation of order l, where $l := m/c$. We assume that the action of f^c is the rotation of angle $2\pi b/l$ in the clockwise direction, where b is a positive integer less than and relatively prime to l. Let q be the least positive integer that satisfies $bq \equiv 1 \bmod l$. We call the irreducible fraction q/l the *valency* of the multiple point x.

We next consider the quotient map $\psi_f : \Sigma \to \overline{\Sigma} := \Sigma/\langle f\rangle$ of Σ under the cyclic group action generated by f. We denote by M_f the set of all the multiple points of f, and set $B_f := \psi_f(M_f)$. Then ψ_f is an m-fold cyclic covering branched at B_f. Say $B_f = \{p_1, p_2, \ldots, p_n\}$, the *branch points* of ψ_f. For each branch point $p_j \in B_f$, $j = 1, 2, \ldots, n$, take a ramification point $\widetilde{p}_j \in \psi_f^{-1}(p_j)$ over p_j. Since $\widetilde{p}_j$ is a multiple point of f, we have the recurrence number c_j and the valency q_j/l_j of $\widetilde{p}_j$. Note that c_j and q_j/l_j do not depend on the choice of a ramification point $\widetilde{p}_j$. The *valency* of the branch point p_j is defined to be q_j/l_j.

The $(n + 2)$-tuple

$$\left(g, m; \frac{q_1}{l_1}, \frac{q_2}{l_2}, \ldots, \frac{q_n}{l_n}\right)$$

is called the *total valency* of f. By the following theorem, the total valency determines a periodic map up to conjugacy.

Theorem 3.1 ([5]). *Let f and f' be two periodic maps of Σ_g with total valencies $(g, m; q_1/l_1, q_2/l_2, \ldots, q_n/l_n)$ and $(g, m'; q'_1/l'_1, q'_2/l'_2, \ldots, q'_{n'}/l'_{n'})$ respectively. Then f is conjugate to f' if and only if the following conditions are satisfied.*

(1) *The orders of f and f' coincide, that is, $m = m'$.*

(2) *the cardinalities of the branch loci B_f and $B_{f'}$ coincide, that is, $n = n'$.*

(3) *For each $j = 1, 2, \ldots, n$, the valencies of branch points p_j and p'_j coincide, that is,*

$$q_j/l_j = q'_j/l'_j,$$

under a successive change of numbering on the branch points if necessary.

Let k denote the genus of $\overline{\Sigma} := \Sigma/\langle f \rangle$ and set $\theta_j := c_j q_j$ for $j = 1, 2, \ldots, n$. In what follows, we use the expression

$$[k, m; \theta_1, \theta_2, \ldots, \theta_n]$$

of the total valency in place of $(k, m; q_1/l_1, q_2/l_2, \ldots, q_n/l_n)$. The following is known. (For example, see [2].)

Proposition 3.2. *Let k be a nonnegative integer, and let m and n be positive integers greater than one. Let $\theta_1, \theta_2, \ldots, \theta_n$ be positive integers less than m. Then there exists a periodic map f of Σ_g with total valency $[k, m; \theta_1, \theta_2, \ldots, \theta_n]$ if and only if the following conditions are satisfied:*

(i) $\theta_1 + \theta_2 + \cdots + \theta_n \equiv 0 \bmod m$.

(ii) *If $k = 0$, then $n \geq 3$ and* $\gcd(m, \theta_1, \theta_2, \ldots, \theta_n) = 1$.

(iii) $2g - 2 = m(2k - 2) + \sum_{j=1}^{n}(m - c_j)$, *where* $c_j := \gcd(m, \theta_j)$.

This proposition immediately yields the following.

Lemma 3.3. *Let k be a nonnegative integer, and let m and n be positive integers greater than one. Let $\theta_1, \theta_2, \ldots, \theta_n$ be positive integers less than m. Then there exists a positive integer g and a periodic map f of an oriented closed real surface Σ of genus g with total valency $[k, m; \theta_1, \theta_2, \ldots, \theta_n]$ if and only if the following conditions are satisfied:*

(I) $\theta_1 + \theta_2 + \cdots + \theta_n \equiv 0 \bmod m$.

(II) *If $k = 0$, then $n \geq 3$ and* $\gcd(m, \theta_1, \theta_2, \ldots, \theta_n) = 1$.

In this case, the genus g of Σ is given by the equation (iii) *in Proposition* 3.2.

The following lemma is clear:

Lemma 3.4. *Let f be a periodic map of an oriented closed real surface $\Sigma = \Sigma_g$ of genus g with total valency $[k, m; \theta_1, \theta_2, \ldots, \theta_n]$. Then the following hold:*

(1) *The order of f^{-1} is m, and the multiple points of f^{-1} coincide with those of f.*

(2) *The quotient space $\Sigma/\langle f^{-1} \rangle$ is identical to $\Sigma/\langle f \rangle$, and the branch points under the quotient map $\psi_{f^{-1}} : \Sigma \to \Sigma/\langle f^{-1} \rangle$ coincide with those under $\psi_f : \Sigma \to \Sigma/\langle f \rangle$.*

(3) *For each branch points p_j, $j = 1, 2, \ldots, n$, if the valency of p_j with respect to f is q_j/l_j, then the valency of p_j with respect to f^{-1} is $(l_j - q_j)/l_j$.*

In particular, the total valency of f^{-1} is given by $[k, m; m-\theta_1, m-\theta_2, \ldots, m-\theta_n]$.

§ 4. Global cyclic quotient construction (general case)

In this section, we will generalize the statement of Proposition 2.5. Let f be a periodic automorphism of a Riemann surface Σ_g of genus $g \geq 1$ and denote its order by $m \geq 2$. We construct a degenerating family of Riemann surfaces of genus g such that the local monodromy automorphisms around the singular fibers correspond to powers of f whose composition is the identity.

First, take another Riemann surface Σ_h of genus $h \geq 1$ and a periodic automorphism μ of Σ_h of order m. We consider an automorphism γ of $\Sigma_g \times \Sigma_h$ given by

$$\gamma \,:\, (x, y) \longmapsto \left(f^{-1}(x), \mu^{-1}(y)\right).$$

Clearly its order is m. The cyclic group G generated by γ acts on $\Sigma_g \times \Sigma_h$. Let $\Phi : \Sigma_g \times \Sigma_h \to \Sigma_h$ denote the second projection of $\Sigma_g \times \Sigma_h$. Then Φ is compatible with the actions of G and $\langle \mu \rangle$, where $\langle \mu \rangle$ denotes the cyclic group generated by μ. In fact, for another generator μ^{-1} of $\langle \mu \rangle$, the following diagram is commutative:

$$\begin{array}{ccc} \Sigma_g \times \Sigma_h & \xrightarrow{\Phi} & \Sigma_h \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\mu^{-1}} \\ \Sigma_g \times \Sigma_h & \xrightarrow{\Phi} & \Sigma_h. \end{array}$$

Then the projection map Φ determines a holomorphic map $\overline{\Phi} : (\Sigma_g \times \Sigma_h)/G \to \Sigma_h/\langle \mu \rangle$. Note that the quotient space $(\Sigma_g \times \Sigma_h)/G$ is a complex surface with (at most) cyclic quotient singularities, while $\overline{\Sigma}_h := \Sigma_h/\langle \mu \rangle$ is a smooth complex curve (that is, a Riemann surface). Let $\mathfrak{r} : M \to (\Sigma_g \times \Sigma_h)/G$ be the minimal resolution map of $(\Sigma_g \times \Sigma_h)/G$. Then we obtain the composition map $\pi := \overline{\Phi} \circ \mathfrak{r} : M \to \Sigma_h/\langle \mu \rangle$.

We will show that the holomorphic map $\pi : M \to \overline{\Sigma}_h$ is a degenerating family of Riemann surfaces of genus g over the Riemann surface $\overline{\Sigma}_h$ such that the local monodromy automorphisms around its singular fibers are powers of f. Recall that the quotient map $\psi_\mu : \Sigma_h \to \overline{\Sigma}_h$ is an m-fold cyclic branched covering. Let $p_1, p_2, \ldots, p_N \in \overline{\Sigma}_h$ be the branch points under ψ_μ. As we see below, they coincide with the singular values of π.

Lemma 4.1. *Let Δ be an open disk contained in $\overline{\Sigma}_h \setminus \{p_1, p_2, \ldots, p_N\}$. Then the restriction $\pi : \pi^{-1}(\Delta) \to \Delta$ is a trivial degeneration of Riemann surfaces of genus g.*

Proof. Noting that Δ does not contain any branch points of the quotient map $\psi_\mu : \Sigma_h \to \overline{\Sigma}_h$, the inverse image $\psi_\mu^{-1}(\Delta)$ is the disjoint union of m open disks in Σ_h, say $\Delta_1, \Delta_2, \ldots, \Delta_m$. Clearly

$$\Phi^{-1}\left(\Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_m\right) = \left(\Sigma_g \times \Delta_1\right) \sqcup \left(\Sigma_g \times \Delta_2\right) \sqcup \cdots \sqcup \left(\Sigma_g \times \Delta_m\right).$$

The action of G permutes $\Sigma_g \times \Delta_1, \Sigma_g \times \Delta_2, \ldots, \Sigma_g \times \Delta_m$, while that of $\langle\mu\rangle$ permutes $\Delta_1, \Delta_2, \ldots, \Delta_m$. Then the diagram

$$\begin{tikzcd}
(\Sigma_g \times \Delta_1) \sqcup (\Sigma_g \times \Delta_2) \sqcup \cdots \sqcup (\Sigma_g \times \Delta_m) \arrow[r, "\Phi"] \arrow[d, "/G"'] & \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_m \arrow[d, "/\langle\mu\rangle"] \\
\Sigma_g \times \Delta \arrow[r, "\overline{\Phi}"] & \Delta
\end{tikzcd}$$

is commutative, which means that $\overline{\Phi}^{-1}(\Delta) = \Sigma_g \times \Delta$ and that the restriction $\overline{\Phi} : \overline{\Phi}^{-1}(\Delta) \to \Delta$ is the natural projection. Since $\Sigma_g \times \Delta$ is smooth, the restriction $\mathfrak{r} : \mathfrak{r}^{-1}(\Sigma_g \times \Delta) \to \Sigma_g \times \Delta$ of the resolution map is trivial. Thus $\pi^{-1}(\Delta)$ $(= \mathfrak{r}^{-1}(\Sigma_g \times \Delta))$ is identical to $\Sigma_g \times \Delta$ and the restriction $\pi : \pi^{-1}(\Delta) \to \Delta$ is the natural projection, which confirms the assertion. $\square$

For each $j = 1, 2, \ldots, N$, let q_j/l_j be the valency of the branch point p_j. We denote the total valency of μ by

$$\left[\overline{h}, m; \theta_1, \theta_2, \ldots, \theta_N\right].$$

Recall that $\overline{h}$ denotes the genus of the Riemann surface $\overline{\Sigma}_h$. We may assume that

$$\frac{q_j}{l_j} = \frac{\theta_j}{m}, \quad j = 1, 2, \ldots, N,$$

if necessary, under a successive change of numbering on the branch points. Setting $c_j := \gcd(m, \theta_j)$, we have $m = c_j l_j$ and $\theta_j = c_j q_j$. Note that c_j is nothing but the recurrence number of a ramification point of p_j under ψ_μ (equivalently, the cardinality of the inverse image $\psi_\mu^{-1}(p_j)$).

Lemma 4.2. *Let Δ be a small disk neighborhood of the branch point p_j in $\overline{\Sigma}_h$. Then the restriction $\pi : \pi^{-1}(\Delta) \to \Delta$ is a degeneration of Riemann surfaces of genus g whose monodromy map coincides with f^{θ_j}.*

Proof. Where c_j is the recurrence number of a ramification point of p_j under $\psi_\mu : \Sigma_h \to \overline{\Sigma}_h$, the inverse image $\psi_\mu^{-1}(\Delta)$ of Δ is the disjoint union of c_j open disks in Σ_h, say $\widetilde{\Delta}_1, \widetilde{\Delta}_2, \ldots, \widetilde{\Delta}_{c_j}$. Clearly

$$\Phi^{-1}\left(\widetilde{\Delta}_1 \sqcup \widetilde{\Delta}_2 \sqcup \cdots \sqcup \widetilde{\Delta}_{c_j}\right) = \left(\Sigma_g \times \widetilde{\Delta}_1\right) \sqcup \left(\Sigma_g \times \widetilde{\Delta}_2\right) \sqcup \cdots \sqcup \left(\Sigma_g \times \widetilde{\Delta}_{c_j}\right).$$

Let b_j be the least positive integer such that $b_j q_j \equiv 1 \bmod l_j$, where q_j/l_j is the valency of p_j (so it is a irreducible fraction). Recall that the cyclic group $\langle\mu^{c_j}\rangle$ generated by μ^{c_j} acts on each disk $\widetilde{\Delta}_i$ as a rotation of angle $2\pi b_j/l_j$ in the clockwise direction. Namely $\mu^{c_j} : \widetilde{\Delta}_i \to \widetilde{\Delta}_i$ is explicitly given by

$$\mu^{c_j}(t) = e^{-2\pi\sqrt{-1}b_j/l_j} t.$$

Note that $\widetilde{\Delta}_1/\langle\mu^{c_j}\rangle, \widetilde{\Delta}_2/\langle\mu^{c_j}\rangle, \ldots, \widetilde{\Delta}_{c_j}/\langle\mu^{c_j}\rangle$ are all identical to Δ. Clearly the quotient group $\langle\mu\rangle/\langle\mu^{c_j}\rangle$ acts as the permutation of $\widetilde{\Delta}_1/\langle\mu^{c_j}\rangle, \widetilde{\Delta}_2/\langle\mu^{c_j}\rangle, \ldots, \widetilde{\Delta}_{c_j}/\langle\mu^{c_j}\rangle$. Similarly, we consider the subgroup G_j of G generated by γ^{c_j}, which acts on each $\Sigma_g \times \widetilde{\Delta}_i$. The action is explicitly given by

$$\begin{aligned}\gamma^{c_j}(x,y) &= \left(f^{-c_j}(x), \mu^{-c_j}(y)\right)\\ &= \left(f^{-c_j}(x), e^{2\pi\sqrt{-1}b_j/l_j} t\right).\end{aligned}$$

On the other hand, the action of the quotient group G/G_j on $\coprod_{i=1}^{c_j}\left((\Sigma_g \times \widetilde{\Delta}_i)/G_j\right)$ is defined as the permutation induced by the G-action. Noting that Φ is compatible to the actions of G_j and $\langle\mu^{c_j}\rangle$, we then obtain the following commutative diagram:

$$\begin{array}{ccc}\coprod_{i=1}^{c_j}\left(\Sigma_g \times \widetilde{\Delta}_i\right) & \xrightarrow{\Phi} & \coprod_{i=1}^{c_j}\widetilde{\Delta}_i\\ {\scriptstyle /G_j}\big\downarrow & & \big\downarrow{\scriptstyle /\langle\mu^{c_j}\rangle}\\ \coprod_{i=1}^{c_j}\left((\Sigma_g \times \widetilde{\Delta}_i)/G_j\right) & & \coprod_{i=1}^{c_j}\widetilde{\Delta}_i/\langle\mu^{c_j}\rangle\\ {\scriptstyle /(G/G_j)}\big\downarrow & & \big\downarrow{\scriptstyle /(\langle\mu\rangle/\langle\mu^{c_j}\rangle)}\\ (\Sigma_g \times \widetilde{\Delta}_i)/G_j & \xrightarrow{\overline{\Phi}} & \Delta.\end{array}$$

Take another generator $\gamma^{c_jq_j}$ of G_j. Since $\theta_j = c_jq_j$ and $b_jq_j \equiv 1 \bmod l_j$, we have

$$\begin{aligned}\gamma^{c_jq_j}(x,y) &= \left(f^{-c_jq_j}(x), e^{2\pi\sqrt{-1}b_jq_j/l_j} t\right)\\ &= \left(f^{-\theta_j}(x), e^{2\pi\sqrt{-1}/l_j} t\right).\end{aligned}$$

Applying Lemma 2.3 to the case that G (resp. γ) is G_j (resp. $\gamma^{c_jq_j}$), we see that the composition map $\pi_j := \overline{\Phi} \circ \mathfrak{r}_j : M_j \to \Delta$ with the resolution map $\mathfrak{r}_j : M_j \to (\Sigma_g \times \widetilde{\Delta}_i)/G_j$ of $(\Sigma_g \times \widetilde{\Delta}_i)/G_j$ is a degeneration of Riemann surfaces of genus g whose monodromy automorphism coincides with f^{θ_j}. (See also Remark below Lemma 2.3.) By construction, $\pi^{-1}(\Delta)$ is identical to M_j and the restriction $\pi : \pi^{-1}(\Delta) \to \Delta$ coincides with $\pi_j : M_j \to \Delta$, which confirms the assertion. □

From Lemma 4.1 and Lemma 4.2, we obtain the following.

Lemma 4.3. *The holomorphic map $\pi : M \to \overline{\Sigma}_h$ is a degenerating family of Riemann surfaces of genus g that has N singular fibers over $p_1, p_2, \ldots, p_N$. Moreover, for the branch point, p_j, $j = 1, 2, \ldots, N$, the local monodromy around the singular fiber over p_j corresponds to f^{θ_j}.*

We now show the main theorem.

Theorem 4.4. *Let $[f]$ be a periodic mapping class of Σ_g and $[f_1], [f_2], \ldots, [f_N]$ ($N \geq 2$) be powers of $[f]$ satisfying $[f_1] \circ [f_2] \circ \cdots \circ [f_N] = 1$. Then there exists a degenerating family of Riemann surfaces of genus g over a Riemann surface of arbitrary genus $k \geq 0$ with N singular fibers whose local monodromies respectively coincides with $[f_1], [f_2], \ldots, [f_N]$ up to conjugacy.*

Proof. By Kerckhoff's theorem, there exists a complex structure on Σ_g and a periodic automorphism f of the Riemann surface Σ_g such that f represents $[f]$. Denote the order of f by m, so $f^m = \mathrm{id}_{\Sigma_g}$. Since $[f_1], [f_2], \ldots, [f_N]$ are powers of $[f]$, for each $j = 1, 2, \ldots, N$, we write as $[f_j] = [f]^{\theta_j}$ where θ_j is a positive integer less than m. In particular, we take a representative $f_j = f^{\theta_j}$ of $[f_j]$.

In what follows, we assume $\gcd(\theta_1, \theta_2, \ldots, \theta_N) = 1$. If $d := \gcd(\theta_1, \theta_2, \ldots, \theta_N) \geq 2$, then we consider an alternative periodic automorphism f^d in place of f. Setting $\theta'_j := \theta_j/d$ and $m' := m/d$, since $f_j = (f^d)^{\theta'_j}$ and $\gcd(\theta'_1, \theta'_2, \ldots, \theta'_N) = 1$, we can also apply the following argument to this case.

Take an arbitrary nonnegative integer k (which will be the genus of the base space of the resulting degenerating family). We first assume that $(N, k) = (2, 0)$. Note that, since $f_1 \circ f_2 = \mathrm{id}_{\Sigma_g}$, we have $f_1 = f$ and $f_2 = f^{-1}$, by permutation of subscripts if necessary. Then, by Lemma 2.4, there exists a degenerating family of Riemann surfaces of genus g over $\mathbb{C}P^1$ with two singular fibers whose local monodromies respectively correspond to f_1 and f_2, which confirms the assertion.

We next consider the case that $(N, k) \neq (2, 0)$. Since $f_1 \circ f_2 \circ \cdots \circ f_N = 1$, the sum $\theta_1 + \theta_2 + \cdots + \theta_N$ is a multiple of m, equivalently,

$$\theta_1 + \theta_2 + \cdots + \theta_N \equiv 0 \bmod m.$$

From Lemma 3.3, there exists a positive integer h and a periodic map μ of an oriented closed real surface Σ_h of genus h such that the valency of μ is

$$[k, m; \theta_1, \theta_2, \ldots, \theta_N].$$

By Kerckhoff's theorem again, we regard Σ_h as a Riemann surface of genus h and μ as a periodic automorphism of Σ_h. We set $\overline{\Sigma}_h := \Sigma_h/\langle\mu\rangle$.

The projection map $\Phi : \Sigma_g \times \Sigma_h \to \Sigma_h$, which is compatible with the G-action, determines a holomorphic map $\overline{\Phi} : (\Sigma_g \times \Sigma_h)/G \to \overline{\Sigma}_h := \Sigma_h/\langle\mu\rangle$. Let $\mathfrak{r} : M \to (\Sigma_g \times \Sigma_h)/G$ be the resolution map of $(\Sigma_g \times \Sigma_h)/G$ that resolves its quotient singularities minimally. we then obtain the composition map $\pi := \overline{\Phi} \circ \mathfrak{r} : M \to \overline{\Sigma}_h$. By Lemma 4.3, we see that $\pi : M \to \overline{\Sigma}_h$ is a degenerating family of Riemann surfaces of genus g that has N singular fibers whose local monodromies correspond to $f^{\theta_1}, f^{\theta_2}, \ldots, f^{\theta_N}$

respectively. Recall that $f_j = f^{\theta_j}$ for each $j = 1, 2, \ldots, N$. Thus π is the desired degenerating family. $\square$

By the similar argument to that for the proof of Theorem 4.4, we can show the following.

Proposition 4.5. *Let f be a periodic automorphism of a complex manifold Z, and let $f_1, f_2, \ldots, f_N$ $(N \geq 2)$ be powers of f satisfying $f_1 \circ f_2 \circ \cdots \circ f_N = \mathrm{id}_Z$. Then there exists a degenerating family of complex manifolds (identical to Z) over a Riemann surface of arbitrary genus with N singular fibers whose local monodromies respectively coincide with $f_1, f_2, \ldots, f_N$.*

§ 5. Patching of two degenerations (core part)

The remaining part of this paper is devoted to describing the total space of the degenerating family $\pi : M \to S$ over the *Riemann sphere* S with *two singular fibers* obtained in Lemma 2.4, that is, we consider the case $(N, k) = (2, 0)$. Recall that, where the Riemann sphere S has the standard open covering $S = \mathbb{C}_0 \cup \mathbb{C}_\infty$, the restriction $\pi : \pi^{-1}(\mathbb{C}_0) \to \mathbb{C}_0$ (resp. $\pi : \pi^{-1}(\mathbb{C}_\infty) \to \mathbb{C}_\infty$) is a (linear) degeneration of Riemann surfaces of genus g whose singular fiber X_0 (resp. X_∞) has the monodromy automorphism f (resp. f^{-1}). To be precise, by using the concept of the total valency again, we will clarify the gluing map $F : \pi^{-1}(\mathbb{C}_0) \setminus X_0 \to \pi^{-1}(\mathbb{C}_\infty) \setminus X_\infty$ of the two linear degenerations, that is, the biholomorphic map that commutes the following diagram:

$$\begin{array}{ccc} \pi^{-1}(\mathbb{C}_0) \setminus X_0 & \xrightarrow{F} & \pi^{-1}(\mathbb{C}_\infty) \setminus X_\infty \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{C}_0 \setminus \{0\} & \xrightarrow{\xi} & \mathbb{C}_\infty \setminus \{\infty\}, \end{array}$$

where $\xi(t) = 1/t$.

For the periodic automorphism f of the Riemann surface $\Sigma = \Sigma_g$ of genus g, denote by $\psi_f : \Sigma \to \overline{\Sigma} := \Sigma/\langle f \rangle$ the quotient map under the cyclic group action generated by f, and let $B_f = \{p_1, p_2, \ldots, p_n\}$ be the set of the branch points of ψ_f. We set

$$\overline{\Sigma}^\times := \overline{\Sigma} \setminus B_f, \quad \Sigma^\times := \Sigma \setminus \psi_f^{-1}(B_f).$$

For each branch point p_j, $j = 1, 2, \ldots, n$, let $\widetilde{p}_{j,1}, \widetilde{p}_{j,2}, \ldots, \widetilde{p}_{j,c_j}$ be the ramification points over p_j under ψ_f, where c_j is the recurrence number of them. Take a small disk Δ_j in $\overline{\Sigma}$ centered at p_j. Then we have $\psi_f^{-1}(\Delta_j) = \widetilde{\Delta}_{j,1} \sqcup \widetilde{\Delta}_{j,2} \sqcup \cdots \sqcup \widetilde{\Delta}_{j,c_j}$, where each $\widetilde{\Delta}_{j,i}$ $(i = 1, 2, \ldots, c_j)$ is an $\langle f^{c_j} \rangle$-invariant open disk in Σ centered at $p_{j,i}$. Note that

$$\overline{\Sigma} = \overline{\Sigma}^\times \cup \left(\bigsqcup_{j=1}^n \Delta_j \right), \quad \Sigma = \Sigma^\times \cup \left(\bigsqcup_{j=1}^n \bigsqcup_{i=1}^{c_j} \widetilde{\Delta}_{j,i} \right).$$

Then we see that $(\Sigma \times S)/G$ consists of two parts as follows:

$$(\Sigma \times S)/G = \left(\Sigma^\times \times S\right)/G \cup \left(\bigsqcup_{j=1}^{n} \bigsqcup_{i=1}^{c_j} \widetilde{\Delta}_{j,i} \times S\right)/G$$

We call $L := \mathfrak{r}^{-1}\left(\left(\Sigma^\times \times S\right)/G\right)$ the *core part* of M, and $N := \mathfrak{r}^{-1}\left(\left(\bigsqcup_{i,j} \widetilde{\Delta}_{j,i} \times S\right)/G\right)$ the *branch part* of M. Since $\left(\Sigma^\times \times S\right)/G$ is smooth, the restriction of $\mathfrak{r}$ to the core part is an isomorphism (so $L \cong \left(\Sigma^\times \times S\right)/G$).

Here we consider a subset

$$L_0 := (\Sigma^\times \times \mathbb{C}_0)/G$$

of $L := (\Sigma^\times \times S)/G$. We see that L_0 is a flat line bundle on $\overline{\Sigma}^\times$ and the bundle projection $\rho : L_0 \to \overline{\Sigma}^\times$ is given by $\rho([x,t]) = [x]$ for $(x,t) \in \Sigma^\times \times \mathbb{C}_0$. Moreover, for each branch point $p_j \in \overline{\Sigma}$ of ψ_f, setting $\Delta_j^\times := \Delta_j \setminus \{p_j\}$, the restriction $L_0|_{\Delta_j^\times}$ of L_0 to $\Delta_j^\times$ is a flat line bundle on $\Delta_j^\times$. Note that $L_0|_{\Delta_j^\times}$ is isomorphic to the trivial bundle $\Delta_j^\times \times \mathbb{C}$, but this bundle isomorphism is not a canonical one. Then, where the valency of p_j is q_j/l_j, the *holonomy* of the flat line bundle $L_0|_{\Delta_j^\times}$ is given by multiplication by $e^{2\pi\sqrt{-1}q_j/l_j}$. In fact, $L_0|_{\Delta_j^\times}$ is given by $\left(\widetilde{\Delta}_{j,i}^\times \times \mathbb{C}\right)/G_j$, where G_j is the subgroup of G generated by

$$\gamma^{c_j} : (x,t) \longmapsto (f^{-c_j}(x), e^{2\pi\sqrt{-1}c_j/m}t).$$

Recall that f^{c_j} acts on $\widetilde{\Delta}_{j,i}^\times$ as the rotation of angle $2\pi b_j/l_j$ in the clockwise direction, where $b_j q_j \equiv 1 \bmod l_j$ holds. Thus

$$\gamma^{c_j}(x,t) = (e^{2\pi\sqrt{-1}b_j/l_j}x, e^{2\pi\sqrt{-1}/l_j}t),$$

and G_j is also generated by

$$\gamma^{c_j q_j} : (x,t) \longmapsto (e^{2\pi\sqrt{-1}/l_j}x, e^{2\pi\sqrt{-1}q_j/l_j}t).$$

Accordingly, we can take such "coordinates" $(\overline{z}, \overline{\zeta})$ of $L_0|_{\Delta_j^\times}$ (where $\overline{z}$ is the base coordinate and $\overline{\zeta}$ is the fiber coordinate) that $\overline{\zeta}$ is multiplied by $e^{2\pi\sqrt{-1}q_j/l_j}$ as $\overline{z}$ goes around p_j once in the counterclockwise direction. These coordinates $(\overline{z}, \overline{\zeta})$ on $L_0|_{\Delta_j^\times}$ are called the *flat coordinates.*

Note that $L_0|_{\Delta_j^\times}$ is smooth. Under identifying $\mathfrak{r}^{-1}(L_0|_{\Delta_j^\times})$ with $L_0|_{\Delta_j^\times}$, the restriction $\pi : L_0|_{\Delta_j^\times} \to \mathbb{C}_0$ of π is given by

$$\pi(\overline{z}, \overline{\zeta}) = \overline{\zeta}^m. \tag{5.1}$$

We thus see that $\pi : L_0|_{\Delta_j^\times} \to \mathbb{C}_0$ is a degeneration with the singular fiber

$$X_0 = m\Delta_j^\times,$$

where m is the multiplicity of $\Delta_j^\times$.

By the same argument as above, setting

$$L_\infty := (\Sigma^\times \times \mathbb{C}_\infty)/G,$$

a subset of $L = (\Sigma \times S)/G$, we see that L_∞ is a flat line bundle on $\overline{\Sigma}^\times$ and that for each branch point $p_j \in \overline{\Sigma}$ of ψ_f, the restriction $L_\infty|_{\Delta_j^\times}$ is a flat line bundle on $\Delta_j^\times$ whose holonomy is is given by multiplication by $e^{2\pi\sqrt{-1}(l_j - q_j)/l_j}$ (by Lemma 3.4). Moreover, the restriction $\pi : \mathfrak{r}^{-1}(L_\infty|_{\Delta_j^\times}) \, (\cong L_\infty|_{\Delta_j^\times}) \to \mathbb{C}_\infty$ is given by

$$\pi(\overline{z}', \overline{\zeta}') = \left(\overline{\zeta}'\right)^m,$$

where $\left(\overline{z}', \overline{\zeta}'\right)$ is the flat coordinates of $L_\infty|_{\Delta_j^\times}$, and it is a degeneration with the singular fiber $X_\infty = m\Delta_j^\times$.

Let us return to the gluing map $F : \pi^{-1}(\mathbb{C}_0) \setminus X_0 \to \pi^{-1}(\mathbb{C}_\infty) \setminus X_\infty$. Here we restrict it to the core part, that is, we consider $F : L_0 \setminus X_0 \to L_\infty \setminus X_\infty$. It is clear that, for each $j = 1, 2, \ldots, n$, $L_0|_{\Delta_j^\times}$ and $L_\infty|_{\Delta_j^\times}$, are patched via F as follows.

Lemma 5.1. *For each $j = 1, 2, \ldots, n$, the restriction $F : L_0|_{\Delta_j^\times} \setminus X_0 \to L_\infty|_{\Delta_j^\times} \setminus X_\infty$ of the gluing map F is given by*

$$F \ : \ \overline{z}' = \overline{z}, \quad \overline{\zeta}' = 1/\overline{\zeta}, \tag{5.2}$$

where $(\overline{z}, \overline{\zeta})$ and $(\overline{z}', \overline{\zeta}')$ are the flat coordinates of $L_0|_{\Delta_j^\times}$ and $L_\infty|_{\Delta_j^\times}$ respectively.

Note that $L_0|_{\Delta_j^\times}$ can be identified with the trivial bundle $\Delta_j^\times \times \mathbb{C}_0$ via

$$\overline{z} = z, \quad \overline{\zeta} = z^{q_j/l_j}\zeta,$$

where $\left(\overline{z}, \overline{\zeta}\right) \in L_0|_{\Delta_j^\times}$ and $(z, \zeta) \in \Delta_j^\times \times \mathbb{C}_0$. Thus, by (5.1), we have

$$\pi(z, \zeta) = z^{\theta_j}\zeta^m, \quad (z, \zeta) \in \Delta_j \times \mathbb{C}_0,$$

since $mq_j/l_j = c_j q_j = \theta_j$. On the other hand, $L_\infty|_{\Delta_j^\times}$ can be identified with the trivial bundle $\Delta_j^\times \times \mathbb{C}_\infty$ via

$$\overline{z}' = z', \quad \overline{\zeta}' = (z')^{(l_j - q_j)/l_j}\zeta',$$

where $(\overline{z}', \overline{\zeta}') \in L_\infty|_{\Delta_j^\times}$ and $(z', \zeta') \in \Delta_j^\times \times \mathbb{C}_\infty$, and we have

$$\pi(z', \zeta') = (z')^{m - \theta_j}(\zeta')^m, \quad (z', \zeta') \in \Delta_j \times \mathbb{C}_\infty.$$

We see that $(z, \zeta) \in \Delta_j^\times \times (\mathbb{C}_0 \setminus \{0\})$ and $(z', \zeta') \in \Delta_j^\times \times (\mathbb{C}_\infty \setminus \{0\})$ are patched by

$$z' = z, \quad \zeta' = z^{-1}\zeta^{-1}.$$

In fact, by (5.2), we have

$$\zeta' = (\overline{z}')^{-(l_j-q_j)/l_j}\overline{\zeta}' = \overline{z}^{-(l_j-q_j)/l_j}\overline{\zeta}^{-1} = z^{-(l_j-q_j)/l_j}\left(z^{q_j/l_j}\zeta\right)^{-1} = z^{-1}\zeta^{-1}.$$

Lemma 5.2. *Under identifying the flat line bundle $L_0|_{\Delta_j^\times}$ (resp. $L_\infty|_{\Delta_j^\times}$) with the trivial bundle $\Delta_j^\times \times \mathbb{C}_0$ (resp. $\Delta_j^\times \times \mathbb{C}_\infty$), the restriction $F : L_0|_{\Delta_j^\times} \setminus X_0 \to L_\infty|_{\Delta_j^\times} \setminus X_\infty$ of the gluing map F is given by*

$$F \,:\, z' = z, \quad \zeta' = z^{-1}\zeta^{-1},$$

where (z,ζ) (resp. (z',ζ')) are the natural coordinates of $\Delta_j^\times \times \mathbb{C}_0$ (resp. $\Delta_j^\times \times \mathbb{C}_\infty$).

§ 6. Patching of two degenerations (branch part), 1

Keep the notations in the previous section. We will reconstruct the branch part N of the top space M of degenerating family $\pi : M \to S$.

First recall that N is the minimal resolution of

$$\left(\bigsqcup_{j=1}^{n}\bigsqcup_{i=1}^{c_j}\widetilde{\Delta}_{j,i}\times S\right)/G = \bigsqcup_{j=1}^{n}\left(\left(\bigsqcup_{i=1}^{c_j}\widetilde{\Delta}_{j,i}\times S\right)/G\right).$$

For each $j = 1, 2, \ldots, n$, we set $N^{(j)} := \mathfrak{r}^{-1}\left(\left(\bigsqcup_{i=1}^{c_j}\widetilde{\Delta}_{j,i}\times S\right)/G\right)$. Then we have

$$N = N^{(1)} \sqcup N^{(2)} \sqcup \cdots \sqcup N^{(n)}.$$

We next consider the subgroup G_j of G generated by γ^{c_j}. Then the quotient group G/G_j of order c_j acts on $\bigsqcup_{i=1}^{c_j}\widetilde{\Delta}_{j,i}\times S$ as the permutation of $\widetilde{\Delta}_{j,1}\times S, \widetilde{\Delta}_{j,2}\times S, \ldots, \widetilde{\Delta}_{j,c_j}\times S$. We then obtain the commutative diagram

$$\begin{array}{ccc}
\bigsqcup_{i=1}^{c_j}\widetilde{\Delta}_{j,i}\times S & \xrightarrow{/(G/G_j)} & \widetilde{\Delta}_j\times S \\
{\scriptstyle /G}\big\downarrow & & \big\downarrow{\scriptstyle /G_j} \\
\left(\bigsqcup_{i=1}^{c_j}\widetilde{\Delta}_{j,i}\times S\right)/G & \xrightarrow{\cong} & \left(\widetilde{\Delta}_j\times S\right)/G_j.
\end{array}$$

Under the identification via this isomorphism, we have $N^{(j)} = \mathfrak{r}^{-1}\left(\left(\widetilde{\Delta}_j\times S\right)/G_j\right)$. Set $N_0^{(j)} := \pi^{-1}(\mathbb{C}_0)\cap N^{(j)}$ and $N_\infty^{(j)} := \pi^{-1}(\mathbb{C}_\infty)\cap N^{(j)}$ so that $N^{(j)} = N_0^{(j)} \cup N_\infty^{(j)}$. Since $\left(\widetilde{\Delta}_j\times S\right)/G_j = \left(\widetilde{\Delta}_j\times \mathbb{C}_0\right)/G_j \cup \left(\widetilde{\Delta}_j\times \mathbb{C}_\infty\right)/G_j$, we have

$$N_0^{(j)} = \mathfrak{r}^{-1}\left(\left(\widetilde{\Delta}_j\times \mathbb{C}_0\right)/G_j\right), \quad N_\infty^{(j)} = \mathfrak{r}^{-1}\left(\left(\widetilde{\Delta}_j\times \mathbb{C}_\infty\right)/G_j\right).$$

The action of G_j on $\widetilde{\Delta}_j\times\mathbb{C}_0$ is given by $\gamma^{c_j}(x,t) = \left(e^{-2\pi\sqrt{-1}b_j/l_j}x, e^{2\pi\sqrt{-1}/l_j}t\right)$. Taking another generator $\gamma^{c_jq_j}$ of G_j, since $\theta_j = c_jq_j$ and $b_jq_j \equiv 1 \bmod l_j$, we have

$$\gamma^{c_jq_j}(x,t) = \left(e^{-2\pi\sqrt{-1}/l_j}x, e^{2\pi\sqrt{-1}q_j/l_j}t\right).$$

We then obtain the minimal resolution $N_0^{(j)}$ of $\left(\widetilde{\Delta}_j \times \mathbb{C}_0\right)/G_j$ by *Hirzebruch-Jung construction* for its cyclic quotient singularity as follows. First note that the irreducible fraction l_j/q_j has the unique continued fraction expansion of negative type

$$\frac{l_j}{q_j} = r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_\lambda}}},$$

where each r_i is an integer greater than one. Take λ copies $\Theta_1, \Theta_2, \ldots, \Theta_\lambda$ of $\mathbb{C}P^1$. For each $i = 1, 2, \ldots, \lambda$, let $\Theta_i = U_i \cup V_i$ be the standard open covering by two copies U_i and V_i of $\mathbb{C}$ with coordinates $w_i \in U_i \setminus \{0\}$ and $z_i \in V_i \setminus \{0\}$ satisfying $z_i = 1/w_i$. Next we construct a line bundle $N_{0,i}$ on Θ_i of order $-r_i$ from $U_i \times \mathbb{C}$ and $V_i \times \mathbb{C}$ by identifying $(z_i, \zeta_i) \in (V_i \setminus \{0\}) \times \mathbb{C}$ with $(w_i, \eta_i) \in (U_i \setminus \{0\}) \times \mathbb{C}$ via

$$\phi_i : z_i = \frac{1}{w_i}, \quad \zeta_i = w_i^{r_i} \eta_i.$$

We then patch $N_{0,i}$ and $N_{0,i+1}$ $(i = 1, 2, \ldots, \lambda - 1)$ by plumbing $\mathsf{pl}_i : V_i \times \mathbb{C} \to U_{i+1} \times \mathbb{C}$ given by $(w_{i+1}, \eta_{i+1}) = (\zeta_i, z_i)$ so that they together define a smooth complex surface, which is nothing but $N_0^{(j)}$.

There exists a unique sequence $(d_0, d_1, \ldots, d_\lambda, d_{\lambda+1})$ of nonnegative integers such that

$$\begin{cases} d_0 = l_j,\ d_1 = q_j,\ d_\lambda = 1,\ d_{\lambda+1} = 0, \\ d_0 > d_1 > \cdots > d_\lambda > d_{\lambda+1}, \\ d_{i-1} + d_{i+1} = r_i d_i,\ i = 1, 2, \ldots, \lambda. \end{cases}$$

We call it the *division sequence* of l_j/q_j. Setting $m_i := c_j d_i$ for each i, we obtain the sequence $m_0, m_1, \ldots, m_\lambda, m_{\lambda+1}$ of integers. Here $m_0 = m$, $m_1 = \theta_j$, $m_\lambda = c_j$ and $m_{\lambda+1} = 0$. Then the restriction $\pi : N_0^{(j)} \to \mathbb{C}_0$ is given by

$$\begin{cases} \pi(w_i, \eta_i) = w_i^{m_{i-1}} \eta_i^{m_i}, & (w_i, \eta_i) \in U_i \times \mathbb{C} \\ \pi(z_i, \zeta_i) = z_i^{m_{i+1}} \zeta_i^{m_i}, & (z_i, \zeta_i) \in V_i \times \mathbb{C} \end{cases}$$

Moreover, we see that $\pi : N_0^{(j)} \to \mathbb{C}_0$ is a degeneration of "disjoint unions of c_j open disks" (identical to $\bigsqcup_{i=1}^{c_j} \widetilde{\Delta}_{j,i}$) with the singular fiber

$$X_0 = m\Delta_j + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda.$$

Before proceeding, we take the two following sequences:

- $(a_0, a_1, \ldots, a_\lambda, a_{\lambda+1})$ is inductively defined by

$$a_0 := 0, \quad a_1 := 1, \quad a_{i+1} := r_i a_i - a_{i-1}\, (i = 1, 2, \ldots, \lambda).$$

- $(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_\lambda, \bar{a}_{\lambda+1})$ is inductively defined by

$$\bar{a}_0 := -1, \quad \bar{a}_1 := 0, \quad \bar{a}_{i+1} := r_i \bar{a}_i - \bar{a}_{i-1}\,(i = 1, 2, \ldots, \lambda).$$

Then we can see that

$$a_\lambda = b_j, \quad a_{\lambda+1} = l_j, \quad \bar{a}_\lambda = \frac{b_j q_j - 1}{l_j}, \quad \bar{a}_{\lambda+1} = q_j. \tag{6.1}$$

By using these sequences, we explicitly describe the patching map between $U_1 \times \mathbb{C}$ and $V_\lambda \times \mathbb{C}$.

Lemma 6.1. *Let $\phi := \phi_\lambda \circ \mathsf{pl}_{\lambda-1} \circ \phi_{\lambda-1} \circ \cdots \circ \mathsf{pl}_1 \circ \phi_1 : U_1^\times \times \mathbb{C}^\times \to V_\lambda^\times \times \mathbb{C}^\times$, be the composition of the patching maps.* (1) *Then ϕ is given by*

$$z_\lambda = w_1^{-b_j} \eta_1^{-(b_j q_j - 1)/l_j}, \quad \zeta_\lambda = w_1^{l_j} \eta_1^{q_j},$$

and (2) *the inverse ϕ^{-1} of ϕ is given by*

$$w_1 = z_\lambda^{-q_j} \zeta_\lambda^{-(b_j q_j - 1)/l_j}, \quad \eta_1 = z_\lambda^{l_j} \zeta_\lambda^{b_j}.$$

Proof. We first claim that $\phi_i \circ \mathsf{pl}_{i-1} \circ \phi_{i-1} \circ \cdots \circ \mathsf{pl}_1 \circ \phi_1 : U_1^\times \times \mathbb{C}^\times \to V_i^\times \times \mathbb{C}^\times$, $i = 1, 2, \ldots, \lambda$, is given by

$$z_i = w_1^{-a_i} \eta_1^{-\bar{a}_i}, \quad \zeta_i = w_1^{a_{i+1}} \eta_1^{\bar{a}_{i+1}}.$$

In fact, we have $z_1 = w_1^{-1} \eta_1^0$ and $\zeta_1 = w_1^{r_1} \eta_1^1$, and supposing the validity of the assertion for $i - 1$, we obtain

$$\begin{aligned}
(z_i, \zeta_i) &= \phi_i \circ \mathsf{pl}_{i-1}(z_{i-1}, \zeta_{i-1}) = \phi_i \circ \mathsf{pl}_{i-1}\left(w_1^{-a_{i-1}} \eta_1^{-\bar{a}_{i-1}}, w_1^{a_i} \eta_1^{\bar{a}_i}\right) \\
&= \phi_i\left(w_1^{a_i} \eta_1^{\bar{a}_i}, w_1^{-a_{i-1}} \eta_1^{-\bar{a}_{i-1}}\right) = \left(\left(w_1^{a_i} \eta_1^{\bar{a}_i}\right)^{-1}, \left(w_1^{a_i} \eta_1^{\bar{a}_i}\right)^{r_i}\left(w_1^{-a_{i-1}} \eta_1^{-\bar{a}_{i-1}}\right)\right) \\
&= \left(w_1^{-a_i} \eta_1^{-\bar{a}_i}, w_1^{r_i a_i - a_{i-1}} \eta_1^{r_i \bar{a}_i - \bar{a}_{i-1}}\right) = \left(w_1^{-a_i} \eta_1^{-\bar{a}_i}, w_1^{a_{i+1}} \eta_1^{\bar{a}_{i+1}}\right).
\end{aligned}$$

In particular, $z_\lambda = w_1^{-a_\lambda} \eta_1^{-\bar{a}_\lambda}, \zeta_\lambda = w_1^{a_{\lambda+1}} \eta_1^{\bar{a}_{\lambda+1}}$, and by (6.1), we have

$$z_\lambda = w_1^{-b_j} \eta_1^{-(b_j q_j - 1)/l_j}, \quad \zeta_\lambda = w_1^{l_j} \eta_1^{q_j},$$

which confirms (1). (2) follows from (1). $\square$

On the other hand, noting that the action of $\gamma^{c_j q_j}$ on $\widetilde{\Delta}_j \times \mathbb{C}_\infty$ is given by

$$\gamma^{c_j q_j}(x, t') = \left(e^{2\pi\sqrt{-1}/l_j} x, e^{2\pi\sqrt{-1}(l_j - q_j)/l_j} t'\right),$$

we see that $\left(\widetilde{\Delta}_j \times \mathbb{C}_\infty\right)/G_j$ also has a cyclic quotient singularity. We then obtain its minimal resolution $N_\infty^{(j)}$ by the same argument as that for $N_0^{(j)}$. In fact, for the continued fraction expansion of negative type

$$\frac{l_j}{l_j - q_j} = s_1 - \cfrac{1}{s_2 - \cfrac{1}{\ddots - \cfrac{1}{s_\nu}}},$$

we construct line bundles $N_{\infty,i}$, $(i = 1, 2, \dots, \nu)$ on Riemann spheres Θ'_i of order $-s_i$ from $U'_i \times \mathbb{C}$ and $V'_i \times \mathbb{C}$ by identifying $(z'_i, \zeta'_i) \in (V'_i \backslash \{0\}) \times \mathbb{C}$ with $(w'_i, \eta'_i) \in (U'_i \backslash \{0\}) \times \mathbb{C}$ via

$$\phi'_i : z'_i = \frac{1}{w'_i}, \quad \zeta'_i = (w'_i)^{s_i} \eta'_i.$$

and patch $N_{\infty,i}$ and $N_{\infty,i+1}$ $(i = 1, 2, \dots, \nu - 1)$ by plumbing $\mathsf{pl}'_i : V'_i \times \mathbb{C} \to U'_{i+1} \times \mathbb{C}$ given by $(w'_{i+1}, \eta'_{i+1}) = (\zeta'_i, z'_i)$ so that we obtain $N_\infty^{(j)}$. Let $(d'_0, d'_1, \dots, d'_\nu, d'_{\nu+1})$ be the division sequence of $l_j/(l_j - q_j)$ and Set $m'_i := c_j d'_i$ for each $i = 0, 1, \dots, \nu + 1$. Note that $m'_0 = m$, $m'_1 = m - \theta_j$, $m'_\nu = c_j$ and $m'_{\nu+1} = 0$. Then the restriction $\pi : N_\infty^{(j)} \to \mathbb{C}_0$ is given by

$$\begin{cases} \pi(w'_i, \eta'_i) = (w')_i^{m'_{i-1}} (\eta'_i)^{m'_i}, & (w'_i, \eta'_i) \in U'_i \times \mathbb{C} \\ \pi(z'_i, \zeta'_i) = (z'_i)^{m'_{i+1}} (\zeta'_i)^{m'_i}, & (z'_i, \zeta'_i) \in V'_i \times \mathbb{C}. \end{cases}$$

We see that $\pi : N_\infty^{(j)} \to \mathbb{C}_\infty$ is a degeneration of disjoint unions of c_j open disks (identical to $\bigsqcup_{i=1}^{c_j} \widetilde{\Delta}_{j,i}$) with the singular fiber

$$X_\infty = m\Delta_j + m'_1\Theta'_1 + m'_2\Theta'_2 + \cdots + m'_\nu\Theta'_\nu.$$

Moreover:

Lemma 6.2. *Let $\phi' := \phi'_\nu \circ \mathsf{pl}'_{\nu-1} \circ \phi'_{\nu-1} \circ \cdots \circ \mathsf{pl}'_1 \circ \phi'_1 : (U'_1)^\times \times \mathbb{C}^\times \to (V'_\nu)^\times \times \mathbb{C}^\times$, be the composition of the patching maps.* (1) *Then ϕ' is given by*

$$z'_\nu = (w'_1)^{-(l_j - b_j)} (\eta'_1)^{-\{(l_j - b_j)(l_j - q_j) - 1\}/l_j}, \quad \zeta'_\nu = (w'_1)^{l_j} (\eta'_1)^{l_j - q_j},$$

and (2) *the inverse ϕ^{-1} of ϕ is given by*

$$w'_1 = (z'_\nu)^{-(l_j - q_j)} (\zeta'_\nu)^{-\{(l_j - b_j)(l_j - q_j) - 1\}/l_j}, \quad \eta'_1 = (z'_\nu)^{l_j} (\zeta'_\nu)^{l_j - b_j}.$$

§ 7. Patching of two degenerations (branch part), 2

We now describe the restriction $F : N_0^{(j)} \setminus X_0 \to N_\infty^{(j)} \setminus X_\infty$ of the gluing map F to each connected component of the branch part.

First recall that

$$N_0^{(j)} = (U_1 \times \mathbb{C}) \cup (V_1 \times \mathbb{C}) \cup \cdots \cup (U_\lambda \times \mathbb{C}) \cup (V_\lambda \times \mathbb{C}).$$

Set $U_i^\times := U_i \setminus \{0\}$ and $V_i^\times := V_i \setminus \{0\}$ for each $i = 1, 2, \ldots, \lambda$. Then $U_i^\times \times \mathbb{C}^\times$ and $V_i^\times \times \mathbb{C}^\times$ are all identified. Therefore,

$$\begin{aligned} N_0^{(j)} \setminus X_0 =& \left(U_1^\times \times \mathbb{C}^\times\right) \cup \left(V_1^\times \times \mathbb{C}^\times\right) \cup \left(U_2^\times \times \mathbb{C}^\times\right) \cup \left(V_2^\times \times \mathbb{C}^\times\right) \cup \cdots \\ & \cup \left(U_\lambda^\times \times \mathbb{C}^\times\right) \cup \left(V_\lambda \times \mathbb{C}^\times\right) \\ =& V_\lambda \times \mathbb{C}^\times. \end{aligned}$$

Similarly we have

$$N_\infty^{(j)} \setminus X_\infty = V'_\nu \times \mathbb{C}^\times.$$

Patching of the core part L_0 and the component $N_0^{(j)}$ of the branch part is given by plumbing $\mathsf{pl}_0 : \Delta_j^\times \times \mathbb{C}_0 \to U_1 \times \mathbb{C}$:

$$(w_1, \eta_1) = (\zeta, z),$$

where the trivial bundle $\Delta_j^\times \times \mathbb{C}_0$ is identified with the flat line bundle $L_0|_{\Delta_j^\times}$. Note that the restriction of π to $U_1 \times \mathbb{C}$ is given by $\pi(w_1, \eta_1) = w_1^{m_0}\eta_1^{m_1} = w_1^m \eta_1^{\theta_j}$, while that to $\Delta_j^\times \times \mathbb{C}_0$ is given by $\pi(z, \zeta) = z^{\theta_j}\zeta^m$. Likewise L_∞ and the component $N_\infty^{(j)}$ of the branch part is given by plumbing $\mathsf{pl}'_0 : \Delta_j^\times \times \mathbb{C}_\infty \to U'_1 \times \mathbb{C}$:

$$(w'_1, \eta'_1) = (\zeta', z'),$$

where the trivial bundle $\Delta_j^\times \times \mathbb{C}_\infty$ is identified with the flat line bundle $L_\infty|_{\Delta_j^\times}$. Now the following necessarily holds.

Lemma 7.1. *The following diagram is commutative.*

$$\begin{array}{ccc} \Delta_j^\times \times \mathbb{C}_0^\times & \xrightarrow{F} & \Delta_j^\times \times \mathbb{C}_\infty^\times \\ {\scriptstyle \mathsf{pl}_0}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{pl}'_0} \\ U_1^\times \times \mathbb{C}^\times & & (U'_1)^\times \times \mathbb{C}^\times \\ {\scriptstyle \phi := \phi_\lambda \circ \mathsf{pl}_{\lambda-1} \circ \phi_{\lambda-1} \circ \cdots \circ \mathsf{pl}_1 \circ \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \phi' := \phi'_\nu \circ \mathsf{pl}'_{\nu-1} \circ \phi'_{\nu-1} \circ \cdots \circ \mathsf{pl}'_1 \circ \phi'_1} \\ V_\lambda^\times \times \mathbb{C}^\times & \xrightarrow{F} & (V'_\nu)^\times \times \mathbb{C}^\times. \end{array}$$

The restriction $F : V_\lambda^\times \times \mathbb{C}^\times \to (V'_\nu)^\times \times \mathbb{C}^\times$ is explicitly given by

$$(z'_\nu, \zeta'_\nu) = \left(\zeta_\lambda z_\lambda, \frac{1}{\zeta_\lambda}\right).$$

In fact,

by Lemmas 5.2, 6.1, 6.2 and 7.1,

$$\begin{aligned}
(z'_\nu, \zeta'_\nu) &= \phi' \circ \mathsf{pl}'_0 \circ F \circ \mathsf{pl}_0^{-1} \circ \phi^{-1}(z_\lambda, \zeta_\lambda) \\
&= \phi' \circ \mathsf{pl}'_0 \circ F \circ \mathsf{pl}_0^{-1} \left(z_\lambda^{-q_j} \zeta_\lambda^{-(b_j q_j - 1)/l_j}, z_\lambda^{l_j} \zeta_\lambda^{b_j} \right) \\
&= \phi' \circ \mathsf{pl}'_0 \circ F \left(z_\lambda^{l_j} \zeta_\lambda^{b_j}, z_\lambda^{-q_j} \zeta_\lambda^{-(b_j q_j - 1)/l_j} \right) \\
&= \phi' \circ \mathsf{pl}'_0 \left(z_\lambda^{l_j} \zeta_\lambda^{b_j}, z_\lambda^{-l_j + q_j} \zeta_\lambda^{-b_j + (b_j q_j - 1)/l_j} \right) \\
&= \phi' \left(z_\lambda^{-l_j + q_j} \zeta_\lambda^{-b_j + (b_j q_j - 1)/l_j}, z_\lambda^{l_j} \zeta_\lambda^{b_j} \right) \\
&= \Bigg(\left(z_\lambda^{-l_j + q_j} \zeta_\lambda^{-b_j + (b_j q_j - 1)/l_j} \right)^{-(l_j - b_j)} \left(z_\lambda^{l_j} \zeta_\lambda^{b_j} \right)^{-\{(l_j - b_j)(l_j - q_j) - 1\}/l_j}, \\
&\qquad \left(z_\lambda^{-l_j + q_j} \zeta_\lambda^{-b_j + (b_j q_j - 1)/l_j} \right)^{l_j} \left(z_\lambda^{l_j} \zeta_\lambda^{b_j} \right)^{l_j - q_j} \Bigg) \\
&= \left(z_\lambda \zeta_\lambda, \zeta_\lambda^{-1} \right).
\end{aligned}$$

Note that

$$\pi(z_\lambda, \zeta_\lambda) = \zeta_\lambda^{c_j} \in \mathbb{C}_0, \quad \pi(z'_\nu, \zeta'_\nu) = \zeta_\nu^{c_j} \in \mathbb{C}_\infty. \tag{7.1}$$

We then have the following.

Lemma 7.2. *The restriction* $F : V_\lambda \times \mathbb{C}^\times \to V'_\nu \times \mathbb{C}^\times$ *of the gluing map* $F : \pi^{-1}(\mathbb{C}_0) \setminus X_0 \to \pi^{-1}(\mathbb{C}_\infty) \setminus X_\infty$ *is given by*

$$(z'_\nu, \zeta'_\nu) = \left(\zeta_\lambda z_\lambda, \frac{1}{\zeta_\lambda} \right). \tag{7.2}$$

By patching $V_\lambda \times \mathbb{C}$ and $V'_\nu \times \mathbb{C}$ via (7.2), we obtain a line bundle $\hat{N}^{(j)}$ on a Riemann sphere Ξ_j of degree -1, where Ξ_j denotes its zero section: $\Xi_j := \{z_\lambda = 0\} \cup \{z'_\nu = 0\}$. Note that $\pi : M \to S$ maps Ξ_j onto S by (7.1). Moreover, if $c_j = 1$, then $\pi : \Xi_j \to S$ is biholomorphic. Namely, Ξ_j is a (-1)-section of π.

Theorem 7.3. *Let $\pi : M \to S$ be the degenerating family of Riemann surfaces over $\mathbb{C}P^1$ with periodic automorphism f obtained in Lemma* 2.4*. Then, if f has n fixed points* (*multiple points with recurrence number* 1)*, then π has n* (-1)*-sections.*

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